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Three-Dimensional Boundary Layer Equations of an Ionized Gas in the Presence of a Strong Magnetic Field

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ABSTRACT

The present study attempts to formulate a new hydrodynamic boundary layer theory for an anisotropic ionized gas. A set of new boundary layer equations is obtained in general orthogonal coordinates. One of the new features indicates that the pressure gradient across the boundary layer can no longer be ignored. Since in the presence of a strong magnetic field the effective thermal conductivity is reduced, study of the heat transfer problem based on the new theory is of interest.

I. INTRODUCTION

The stress tensor of a "simple" ionized gas (a gas composed of identical charged particles) in the presence of a magnetic field has previously been demonstrated by Chapman and Cowling (Ref. 1). If ω represents the cyclotron frequency and τ the mean collision time of the charged particles, then the components of the viscous stress tensor take the following form:

$$\tau_{11} = -2\mu\epsilon_{11}$$

$$\tau_{22} = -\frac{2\mu}{1+\frac{16}{9}\omega^2\tau^2}\left[\epsilon_{22} + \frac{1}{2}(\epsilon_{22}+\epsilon_{33})\frac{16}{9}\omega^2\tau^2 + \frac{4}{3}\omega\tau\epsilon_{23}\right]$$

$$\tau_{33} = -\frac{2\mu}{1+\frac{16}{9}\omega^2\tau^2}\left[\epsilon_{33} + \frac{1}{2}(\epsilon_{22}+\epsilon_{33})\frac{16}{9}\omega^2\tau^2 - \frac{4}{3}\omega\tau\epsilon_{23}\right]$$

$$\tau_{23} = \tau_{32} = -\frac{2\mu}{1+\frac{16}{9}\omega^2\tau^2}\left[\epsilon_{23} + \frac{1}{2}(\epsilon_{33}-\epsilon_{22})\frac{4}{3}\omega\tau\right]$$

$$\tau_{12} = \tau_{21} = -\frac{2\mu}{1+\frac{4}{9}\omega^2\tau^2}\left[\epsilon_{12} + \frac{2}{3}\omega\tau\epsilon_{13}\right]$$

$$\tau_{13} = \tau_{31} = -\frac{2\mu}{1+\frac{4}{9}\omega^2\tau^2}\left[\epsilon_{13} - \frac{2}{3}\omega\tau\epsilon_{12}\right]$$

where μ is the viscosity,

$$\epsilon_{ij} = \left(\frac{1}{2} \quad \frac{1}{h_i}\frac{\partial u_j}{\partial x_i} + \frac{1}{h_j}\frac{\partial u_i}{\partial x_j} + \Gamma_{ijk}u_k + \Gamma_{jik}u_k\right) - \frac{1}{3}\,\delta_{ij}\ \mathrm{div}\ \mathbf{v}$$

x_1, x_2, and x_3 are the three orthogonal coordinates,[1] u_1, u_2, and u_3 are the corresponding velocity components, and Γ_{ijk}, the Christoffel symbol, is defined as

$$\Gamma_{ijk} = \frac{1}{h_j h_k}\left(\frac{\partial h_j}{\partial x_k}\,\delta_{ij} - \frac{\partial h_k}{\partial x_j}\,\delta_{ik}\right)$$

The derivation of the above stress components is based on the postulation that the homogenous magnetic field is in the direction x_1. A few conclusions can be drawn immediately. First, the stress component parallel to the magnetic field is not affected. Second, if the magnetic field is sufficiently strong, the cyclotron frequency becomes larger than the mean collision frequency, and the conventional "isotropy" assumption breaks down. In this case, the viscosity is no longer a scalar quantity, and the Navier-Stokes stress tensor in classical hydrodynamics must be modified. Third, the effective viscosities in the direction transverse to the magnetic field lines are reduced by a factor $1 + \alpha\omega^2\tau^2$, where α is a constant coefficient.

On the other hand, if the heat conduction phenomenon in the same gas is studied more closely, it can be seen that thermal conductivity also becomes a tensorial quantity. In this case the effective conductivity in the direction normal to the magnetic field is reduced by a factor $1 + \omega^2\tau^2$; in addition, the Righi-Leduc effect appears (Ref. 1). If $\mathbf{Q}_{||}$ represents the heat flux parallel to the magnetic field, and $\mathbf{Q}_{\perp}$ the heat flux vector normal to the field, then

$$\mathbf{Q}_{||} = -k\nabla_{||}T$$

$$\mathbf{Q}_{\perp} = -\frac{k}{1+\omega^2\tau^2}\left(\nabla_{\perp}T + \omega\tau B^{-1}\,\mathbf{B}\times\nabla_{\perp}T\right)$$

Although all these results are obtained from the very simple gas model which assumes identical charged particles, they are, nevertheless, of considerable significance. It may be expected that, in general, the presence of electrons and ions in a gas flow will cause the hydrodynamic

[1]In the following discussions, the general orthogonal coordinate system is used.

characteristics and transport properties of the gas to be affected by the magnetic field. This effect is different from that customarily discussed in the classical theory of hydromagnetics, in which the only interaction mechanism considered is due to the presence of ponderomotive force, or what may be called the "resultant" Lorentz force. In most of the existing solutions of magnetogasdynamics the transport properties of the gas are postulated to be unaffected by the magnetic field. Obviously, the old concept needs some modification when $\omega^2\tau^2 \geq 1$. This is the purpose of the present investigation.

It is expected that this more general theory may have important applications to practical problems. For example, one of the most interesting subjects in the past few years has been the attempt to achieve magnetohydrodynamic cooling, that is, utilization of the principle of magnetohydrodynamic interaction to reduce the heat transfer in a flow system. This is of extreme importance to the re-entry problem and to many other problems in which hot ionized gases may be present and heat transfer at a solid-gas interface is serious. The old concept is that the imposed magnetic field may modify the flow field. Since the temperature distribution depends upon the flow field, it would be expected that the temperature field would be affected also by the externally imposed magnetic field. The principal objective of this approach is to select the field so that the temperature gradient near the solid-gas interface is reduced. In an attempt to study this problem, some authors (Ref. 2) have introduced the Reynolds analogy in the boundary layer theory and have suggested that the magnetic field be imposed in such a way that the skin friction is reduced. Using this model (Ref. 3-5) as a basis a number of calculations have been obtained. The results have shown that the heat transfer rate may be reduced by only a small fraction, substantially less than desired or expected. However, it is not apparent that the Reynolds analogy applies to these systems. Thus conclusions based on this concept may not be meaningful.

The original Reynolds analogy is based on the relation (Ref. 6)

$$\left(\frac{\partial T}{\partial y}\right)_w \approx (Pr)^{\frac{1}{2}} \frac{T_w - T_\infty}{u_\infty} \left(\frac{\partial u}{\partial y}\right)_w$$

where $(\partial T/\partial y)_y$ and $(\partial u/\partial y)_w$ are, respectively, the temperature and velocity gradients at the wall, u is the velocity component parallel to the wall, y is the coordinate normal to the wall, and Pr is the Prandtl number. The subscript w indicates quantities at the wall, and ∞ indicates quantities in the free stream or at the outer edge of the viscous boundary layer. Now, if the subscript m is introduced to indicate quantities after the magnetic field is imposed,

$$\left(\frac{\partial T}{\partial y}\right)_{wm} \approx (Pr_m)^{\frac{1}{2}} \frac{T_{wm} - T_{\infty m}}{u_{\infty m}} \left(\frac{\partial u}{\partial y}\right)_{wm}$$

Thus

$$\frac{\left(\frac{\partial T}{\partial y}\right)_w}{\left(\frac{\partial T}{\partial y}\right)_{wm}} \approx \left(\frac{Pr}{Pr_m}\right)^{\frac{1}{2}} \frac{(T_w - T_\infty)u_{\infty m}}{(T_{wm} - T_{\infty m})u_\infty} \frac{\left(\frac{\partial u}{\partial y}\right)_w}{\left(\frac{\partial u}{\partial y}\right)_{wm}}$$

If it is postulated that transport properties are not affected by the magnetic field,

$$Pr = Pr_m$$

If the skin friction is reduced,

$$\left(\frac{\partial u}{\partial y}\right)_{wm} < \left(\frac{\partial u}{\partial y}\right)_w$$

Also, it is generally expected[2] that $u_\infty > u_{\infty m}$. Therefore, even if $T_w - T_\infty$ remains constant before and after the magnetic field is applied, it appears quite indefinite whether or not $(\partial T/\partial y)_{wm}$ will be smaller than $(\partial T/\partial y)_w$.

In order to suggest a new concept for the magnetohydrodynamic cooling technique, it is possible to impose the magnetic field in such a way that both the viscosity and thermal conductivity may be reduced. To investigate the feasibility of this idea is part of the primary motivation of the present study. Another point of interest is that the anisotropic properties of the plasma and the new behavior of the viscous friction may give rise to some new hydrodynamic feature, especially in the three-dimensional case. The ultimate purpose of this paper is to furnish an introductory study of these items.

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[2]Because the velocity at the outer edge of the boundary layer may also be significantly distorted and the magnitude is reduced.

II. GENERAL THEORY

A. Modified Navier-Stokes Stress Tensor

In Section I mention was made of the form of the stress tensor derived by Chapman and Cowling for the "simple" gas. Further discussion along these lines has been given recently by Burgers (Ref. 7) for the case of a fully ionized gas (plasma). In the latter work, the effective transverse viscosities in the stress tensor appear to be more complicated than those in the "simple" gas. The components of the viscous stress tensor now take the form:

$$\tau_{11} = -\,2\mu\left(1 + \frac{\sqrt{\gamma}}{1+\sqrt{2}}\right)\epsilon_{11}$$

$$\tau_{22} = -\,\mu\left[(\epsilon_{22}-\epsilon_{33}) + \frac{1}{D_1}(\epsilon_{22}-\epsilon_{33})\left(1 + 4\,\frac{\omega^2\tau^2}{(1+\sqrt{2})^2}\right) + \frac{4}{D_1}\,\epsilon_{23}\,\omega\tau\sqrt{\gamma}\left(1 + 4\,\frac{\omega^2\tau^2}{(1+\sqrt{2})^2}\right)\right]$$

$$-\,\mu\left\{\frac{\sqrt{\gamma}}{1+\sqrt{2}}(\epsilon_{22}+\epsilon_{33}) + \frac{1}{D_1}\left[(\epsilon_{22}-\epsilon_{33})\,\frac{\sqrt{\gamma}}{1+\sqrt{2}}\quad 1 + 4\,\frac{\omega^2\tau^2}{(1+\sqrt{2})^2}\,\gamma\right.\right.$$

$$\left.\left.-\,\epsilon_{23}\,\frac{4\omega\tau\sqrt{\gamma}}{(1+\sqrt{2})^2}\,(1 + 4\omega^2\tau^2\gamma)\right]\right\}$$

$$\tau_{33} = -\,\mu\left[(\epsilon_{22}+\epsilon_{33}) - \frac{1}{D_1}(\epsilon_{22}-\epsilon_{33})\left(1 + 4\,\frac{\omega^2\tau^2}{(1+\sqrt{2})^2}\right) - \frac{4}{D}\,\epsilon_{23}\,\omega\tau\sqrt{\gamma}\left(1 + 4\,\frac{\omega^2\tau^2}{(1+\sqrt{2})^2}\right)\right]$$

$$-\,\mu\left\{\frac{\sqrt{\gamma}}{1+\sqrt{2}}(\epsilon_{22}+\epsilon_{33}) - \frac{1}{D_1}\left[(\epsilon_{22}-\epsilon_{33})\,\frac{\sqrt{\gamma}}{1+\sqrt{2}}\left(1 + 4\omega^2\tau^2\,\frac{\gamma}{(1+\sqrt{2})^2}\right)\right.\right.$$

$$\left.\left.-\,\epsilon_{23}\,\frac{4\omega\tau\sqrt{\gamma}}{(1+\sqrt{2})^2}\,(1 + 4\omega^2\tau^2\gamma)\right]\right\}$$

$$\tau_{23} = -\,\frac{2\mu}{D_1}\left\{\left[-\,(\epsilon_{22}-\epsilon_{33})\,\omega\tau\sqrt{\gamma}\left(1 + 4\,\frac{\omega^2\tau^2}{(1+\sqrt{2})^2}\right) + \epsilon_{23}\left(1 + 4\,\frac{\omega^2\tau^2}{(1+\sqrt{2})^2}\right)\right.\right.$$

$$\left.\left.+\,4(\epsilon_{22}-\epsilon_{33})\,\frac{\omega\tau\sqrt{\gamma}}{(1+\sqrt{2})^2}\,(1 + 4\omega^2\tau^2\gamma) + \epsilon_{23}\,\frac{\sqrt{\gamma}}{1+\sqrt{2}}\,(1 + 4\omega^2\tau^2\gamma)\right]\right\}$$

where

$$D_1 = \left(1 + 4\,\frac{\omega^2\tau^2}{(1+\sqrt{2})^2}\right)(1 + 4\omega^2\tau^2\gamma)$$

ω = electron cyclotron frequency

τ = electron collision time

γ = electron mass/ion mass

and

$$\tau_{13} = -\frac{2\mu}{D}\left\{\left[\frac{\omega\tau\sqrt{\gamma}}{(1+\sqrt{2})^2}(1+\omega^2\tau^2\gamma) - \omega\tau\sqrt{\gamma}\left(1 + \frac{\omega^2\tau^2}{1+\sqrt{2}}\right)\right]\epsilon_{12} + \left[\left(1 + \frac{\omega^2\tau^2}{(1+\sqrt{2})^2}\right) + \frac{\sqrt{\gamma}}{1+\sqrt{2}}(1+\omega^2\tau^2\gamma)\right]\epsilon_{13}\right\} = \epsilon_{31}$$

with

$$D = \left(1 + \frac{\omega^2\tau^2}{(1+\sqrt{2})^2}\right)(1+\omega^2\tau^2\gamma)$$

$$\tau_{12} = -\frac{2\mu}{D}\left\{\left[\left(1 + \omega^2\tau^2\,\frac{1}{(1+\sqrt{2})^2}\right) + \frac{\sqrt{\gamma}}{1+\sqrt{2}}(1+\omega^2\tau^2\gamma)\right]\epsilon_{12} + \left[\omega\tau\sqrt{\gamma}\left(1 + \frac{\omega^2\tau^2}{(1+\sqrt{2})^2}\right) - \frac{\omega\tau\sqrt{\gamma}}{(1+\sqrt{2})^2}(1+\omega^2\tau^2\gamma)\right]\epsilon_{13}\right\} = \tau_{21}$$

For simplicity τ_{22} and τ_{33} can be rewritten as follows:

$$\tau_{22} = -\frac{2\mu}{D_1}\left\{\left[\left(1 + \frac{\sqrt{\gamma}}{1+\sqrt{2}}\right) + 4\,\frac{\omega^2\tau^2}{(1+\sqrt{2})^2}\right]\epsilon_{22}\right.$$

$$+\frac{1}{2}(\epsilon_{22}+\epsilon_{33})\left[\left(1 + \frac{\sqrt{\gamma}}{1+\sqrt{2}}\right)4\omega^2\tau^2\gamma\left(1 + 4\,\frac{\omega^2\tau^2}{(1+\sqrt{2})^2}\right) + \frac{4\sqrt{\gamma}}{(1+\sqrt{2})^2}\,\omega^2\tau^2\right]$$

$$\left.+\frac{1}{2}\epsilon_{23}\left[4\omega\tau\sqrt{\gamma}\left(1 + \frac{4\omega^2\tau^2}{(1+\sqrt{2})^2}\right) - \frac{4\omega\tau\sqrt{\gamma}}{(1+\sqrt{2})^2}\,(1 + 4\omega^2\tau^2\gamma)\right]\right\}$$

$$\tau_{33} = -\frac{2\mu}{D_1}\left\{\left[\left(1 + \frac{\sqrt{\gamma}}{1+\sqrt{2}}\right) + 4\,\frac{\omega^2\tau^2}{(1+\sqrt{2})^2}\right]\epsilon_{33}\right.$$

$$+\frac{1}{2}(\epsilon_{22}+\epsilon_{33})\left[\left(1 + \frac{\sqrt{\gamma}}{1+\sqrt{2}}\right)4\omega^2\tau^2\gamma\left(1 + \frac{4\omega^2\tau^2}{(1+\sqrt{2})^2}\right) + \frac{4\sqrt{\gamma}}{(1+\sqrt{2})^2}\,\omega^2\tau^2\right]$$

$$\left.-\frac{1}{2}\epsilon_{23}\left[4\omega\tau\sqrt{\gamma}\left(1 + \frac{4\omega^2\tau^2}{(1+\sqrt{2})^2}\right) - \frac{4\omega\tau\sqrt{\gamma}}{(1+\sqrt{2})^2}\,(1 + 4\omega^2\tau^2\gamma)\right]\right\}$$

Furthermore, the following notation is employed:

$$\Lambda_1 \equiv 1 + \frac{\sqrt{\gamma}}{1+\sqrt{2}} \approx 1$$

$$\Lambda_2 \equiv 1 + \frac{\sqrt{\gamma}}{1+\sqrt{2}} + \frac{\omega^2\tau^2}{(1+\sqrt{2})^2} \approx 1 + \frac{\omega^2\tau^2}{(1+\sqrt{2})^2}$$

$$\Lambda_3 \equiv \left(1 + \frac{\sqrt{\gamma}}{1+\sqrt{2}}\right)4\omega^2\tau^2\gamma\left(1 + \frac{4\omega^2\tau^2}{(1+\sqrt{2})^2}\right) + \sqrt{\gamma}\,\frac{4\omega^2\tau^2}{(1+\sqrt{2})^2}$$

$$\Lambda_4 \equiv 4\omega\tau\sqrt{\gamma}\left(1 + 4\omega^2\tau^2 \frac{1}{(1+\sqrt{2})^2}\right) - \frac{4\omega\tau\sqrt{\gamma}}{(1+\sqrt{2})^2}(1 + 4\omega^2\tau^2\gamma)$$

$$\Lambda_5 \equiv \left(1 + 4\frac{\omega^2\tau^2}{(1+\sqrt{2})^2}\right) + \frac{\sqrt{\gamma}}{1+\sqrt{2}}(1 + 4\omega^2\tau^2\gamma)$$

$$\Lambda_6 \equiv \left(1 + \frac{\omega^2\tau^2}{(1+\sqrt{2})^2}\right) + \frac{\sqrt{\gamma}}{1+\sqrt{2}}(1 + \omega^2\tau^2\gamma)$$

$$\Lambda_7 \equiv \omega\tau\sqrt{\gamma}\left(1 + \frac{\omega^2\tau^2}{(1+\sqrt{2})^2}\right) - \frac{\omega\tau\sqrt{\gamma}}{(1+\sqrt{2})^2}(1 + \omega^2\tau^2\gamma)$$

Hence,

$$\begin{aligned}
\tau_{11} &= -2\mu\Lambda_1\epsilon_{11} = 2\mu\epsilon_{11} \\
\tau_{22} &= -\frac{2\mu}{D_1}\left[\Lambda_2\epsilon_{22} + \frac{\Lambda_3}{2}(\epsilon_{22}+\epsilon_{33}) + \frac{\Lambda_4}{2}\epsilon_{23}\right] \\
\tau_{33} &= -\frac{2\mu}{D_1}\left[\Lambda_2\epsilon_{33} + \frac{\Lambda_3}{2}(\epsilon_{22}+\epsilon_{33}) - \frac{\Lambda_4}{2}\epsilon_{23}\right] \\
\tau_{23} &= -\frac{2\mu}{D_1}\left[\Lambda_5\epsilon_{23} + \Lambda_4(\epsilon_{22}-\epsilon_{33})\right] = \tau_{32} \\
\tau_{12} &= -\frac{2\mu}{D}(\Lambda_6\epsilon_{12} + \Lambda_7\epsilon_{13}) = \tau_{21} \\
\tau_{13} &= -\frac{2\mu}{D}(\Lambda_6\epsilon_{13} - \Lambda_7\epsilon_{12}) = \tau_{31}
\end{aligned} \tag{1}$$

In the derivation of the above results, γ has been considered to be small compared to unity.

It is of interest to note that the functional forms of these stress components are comparable with those in the simple gas case. Further extension of these results to partially ionized gas has not been obtained. However, it may be postulated that the presence of neutral particles may not affect the functional forms of the stress tensor because the magnetic field has no direct influence on the motion of the neutral particles. Nevertheless, in that case, the transverse viscosities will be definitely affected because of the different collision processes, and, from this, it can be postulated that Eq. (1) is the general form of the stress tensor; the transverse viscosities can be determined later.

The corresponding viscous dissipation function of this new viscous stress tensor can now be discussed. By definition, the dissipation function is written as

$$\Phi = \tau_{ij} \left[\frac{1}{h_i} \frac{\partial u_j}{\partial x_i} + \Gamma_{ijk} u_k \right]$$

The orthogonal coordinates mentioned in Section I are used here. For simplicity, the following notation is employed:

$$S_{ij} = \frac{1}{h_j} \frac{\partial u_i}{\partial x_j} + \Gamma_{ijk} u_k$$

and

$$e_{ij} = \frac{1}{2} [S_{ij} + S_{ji}] , \quad e_{ii} = S_{ii}$$

Thus

$$\epsilon_{ij} = e_{ij} - \frac{1}{3} \delta_{ij} \operatorname{div} \mathbf{v}$$

$$= \frac{1}{2} [S_{ij} + S_{ji}] - \delta_{ij} \frac{1}{3} \operatorname{div} \mathbf{v}$$

Then,

$$\Phi = 2\mu\Lambda_1\left(e_{11}^2 - \frac{e_{11}}{3}\,\text{div}\,\mathbf{v}\right) + \frac{2\mu}{D_1}\left[\Lambda_2 e_{22}^2 + \frac{\Lambda_3}{2}(e_{22}^2 - e_{22}e_{33})\right.$$

$$\left. - \frac{e_{22}}{3}(\Lambda_2 + \Lambda_3)\,\text{div}\,\mathbf{v} + \frac{\Lambda_4}{2}(e_{23}e_{22})\right] + \frac{2\mu}{D_1}\left[\Lambda_2 e_{33}^2 + \frac{\Lambda_3}{2}(e_{33}^2 + e_{22}e_{33})\right.$$

$$\left. - \frac{e_{33}}{3}(\Lambda_2 + \Lambda_3)\,\text{div}\,\mathbf{v} - \frac{\Lambda_4}{2}(e_{23}e_{33})\right] + \frac{2\mu}{D_1}\left[2\Lambda_5(e_{23})^2 - \frac{\Lambda_4}{2}(e_{22} - e_{33})\,e_{23}\right]$$

$$+ \frac{2\mu}{D}\left[2\Lambda_6(e_{12})^2 + 2\Lambda_7 e_{21}e_{31}\right] + \frac{2\mu}{D}\left[2\Lambda_6(e_{13})^2 - 2\Lambda_7 e_{21}e_{31}\right]$$

$$= 2\mu\Lambda_1 e_{11}^2 + 2\mu\frac{\Lambda_2}{D}(e_{22}^2 + e_{33}^2) + \frac{\mu\Lambda_3}{D_1}(e_{22} + e_{33})^2$$

$$+ 4\mu\left[\frac{\Lambda_5}{D_1}e_{23}^2 + \frac{\Lambda_6}{D}(e_{12}^2 + e_{13}^2)\right] - \frac{2\mu}{3}\left[\Lambda_1 e_{11} + \frac{(\Lambda_2 + \Lambda_3)}{D_1}(e_{22} + e_{33})\right]\text{div}\,\mathbf{v}$$

Since $\gamma << 1$ and $D_1\Lambda_1 \approx \Lambda_2 + \Lambda_3$,

$$\Phi = 2\mu e_{11}^2 + \frac{2\mu}{D_1}\Lambda_2(e_{22}^2 + e_{33}^2) + \frac{\mu}{D_1}\Lambda_3(e_{22} + e_{33})^2$$

$$+ 4\mu\left[\frac{\Lambda_5}{D_1}e_{23}^2 + \frac{\Lambda_6}{D}(e_{12}^2 + e_{13}^2)\right] - \frac{2}{3}\mu(e_{11} + e_{22} + e_{33})^2 \qquad (2)$$

For the nonmagnetic case, it is known that

$$(\Phi)_{\text{nonmagnetic}} = 2\mu \left[e_{11}^2 + e_{22}^2 + e_{33}^2 + 2(e_{12}^2 + e_{23}^2 + e_{31}^2) - \frac{1}{3}(e_{11} + e_{22} + e_{33})^2 \right]$$

Hence the dissipation function in the present case can be written as

$$\Phi = (\Phi)_{\text{nonmagnetic}} - \left\{ \frac{\mu}{D_1} \Lambda_3 (e_{22} - e_{33})^2 + 4\mu \left[\left(1 - \frac{\Lambda_5}{D_1}\right) e_{23}^2 + \left(1 - \frac{\Lambda_6}{D}\right)(e_{12}^2 + e_{13}^2) \right] \right\} \tag{3}$$

where Φ cannot be negative. However, this does not mean that the viscous dissipation in the magnetic case is less than that in the nonmagnetic case.

B. Boundary Layer Approximation

It was noted previously that when the magnetic field is sufficiently strong the viscous stress tensor appearing in the Navier-Stokes equations has to be modified. It appears at first that such a modification does not support any hope for immediate application because the mathematical solution of such a complicated system of differential equations is almost out of the question. The difficulties arise not only from the complicated form of the stress tensor but also from the fact that the transverse viscosities in general are functions of composition of the ionized gas, temperature, pressure, magnetic field strength, etc. Thus, the governing equations of momentum and energy are highly nonlinear. However, upon closer study it is seen that the applicability of this new theory is not entirely denied by these apparent mathematical complexities. In fact, a resolution can be found which is similar to the classical boundary layer approximation given by Prandtl, since in most gasdynamic problems the viscous Reynolds number is large. This approximation introduces considerable simplification in the differential equations: not only the stress tensor but also other physical quantities are greatly simplified, as will be shown later.

In the subsequent discussions, interest is restricted to the case wherein the imposed magnetic field is parallel to the gas–solid interface. This case is important because the thermal conductivity may be reduced in the direction normal to the wall. Furthermore, the electrical conductivity is considered finite and all the viscosities are considered to have the same order of magnitude.

In order to discuss the boundary layer approximation, the coordinate system to be used is defined. In general, it is desirable to adopt such orthogonal coordinates as to avoid unnecessary complexities. Hereafter, x_1 and x_2 are considered as the surface coordinates, and

the surface coordinate system is determined by the metric

$$ds^2 = h_1 dx_1^2 + h_2 dx_2^2$$

The complete coordinate system used to describe the fluid motion should include one additional coordinate x_3 in the direction normal to the surface. For this system the complete metric becomes

$$ds^2 = h_1 dx_1^2 + h_2 dx_2^2 + h_3 dx_3^2$$

where, in the most general case, h_1, h_2, and h_3 may be functions of x_1, x_2, and x_3.

If the corresponding components of the velocity vector $\mathbf{v}$ are denoted by u_1, u_2, and u_3, then, from the boundary layer theory,

$$u_1 \sim O(1) \qquad x_1 \sim O(1)$$

$$u_2 \sim O(1) \qquad x_2 \sim O(1)$$

$$u_3 \sim O(\delta) \qquad x_3 \sim O(\delta)$$

where

$$\delta << 1$$

Again,

$$\epsilon_{11} = \left(\frac{1}{h_1} \frac{\partial u_1}{\partial x_1} + \frac{u_2}{h_1 h_2} \frac{\partial h_1}{\partial x_2} + \frac{u_3}{h_1 h_3} \frac{\partial h_1}{\partial x_3} \right) - \frac{1}{3} \operatorname{div} \mathbf{v}$$

$$\epsilon_{22} = \left(\frac{1}{h_1} \frac{\partial u_2}{\partial x_2} + \frac{u_1}{h_2 h_1} \frac{\partial h_2}{\partial x_1} + \frac{u_3}{h_2 h_3} \frac{\partial h_2}{\partial x_3} \right) - \frac{1}{3} \operatorname{div} \mathbf{v}$$

$$\epsilon_{33} = \left(\frac{1}{h_3} \frac{\partial u_3}{\partial x_3} + \frac{u_1}{h_1 h_3} \frac{\partial h_3}{\partial x_1} + \frac{u_2}{h_2 h_3} \frac{\partial h_3}{\partial x_2} \right) - \frac{1}{3} \operatorname{div} \mathbf{v}$$

$$\epsilon_{12} = \frac{1}{2}\left[\left(\frac{1}{h_1}\frac{\partial u_2}{\partial x_1} + \frac{1}{h_2}\frac{\partial u_1}{\partial x_2}\right) - \left(\frac{u_2}{h_2 h_1}\frac{\partial h_2}{\partial x_1} + \frac{u_1}{h_1 h_2}\frac{\partial h_1}{\partial x_2}\right)\right]$$

$$\epsilon_{13} = \frac{1}{2}\left[\left(\frac{1}{h_1}\frac{\partial u_3}{\partial x_1} + \frac{1}{h_3}\frac{\partial u_1}{\partial x_3}\right) - \left(\frac{u_3}{h_3 h_1}\frac{\partial h_3}{\partial x_1} + \frac{u_1}{h_1 h_3}\frac{\partial h_1}{\partial x_3}\right)\right]$$

$$\epsilon_{23} = \frac{1}{2}\left[\left(\frac{1}{h_2}\frac{\partial u_3}{\partial x_2} + \frac{1}{h_3}\frac{\partial u_2}{\partial x_3}\right) - \left(\frac{u_3}{h_2 h_3}\frac{\partial h_3}{\partial x_2} + \frac{u_2}{h_2 h_3}\frac{\partial h_2}{\partial x_3}\right)\right]$$

In the boundary layer case, it may be postulated that h_1 and h_2 weakly depend upon x_3. If it is desired to retain only quantities of order $O(1/\delta)$ and neglect quantities of $O(1)$, the following may be written:

$$\epsilon_{11} \sim O(1)$$

$$\epsilon_{22} \sim O(1)$$

$$\epsilon_{33} \sim O(1)$$

$$\epsilon_{12} \sim O(1)$$

$$\epsilon_{13} \approx \frac{1}{2}\frac{1}{h_3}\frac{\partial u_1}{\partial x_3} \sim O\left(\frac{1}{\delta}\right)$$

$$\epsilon_{23} \approx \frac{1}{2}\frac{1}{h_3}\frac{\partial u_2}{\partial x_3} \sim O\left(\frac{1}{\delta}\right)$$

and

$$\tau_{11} \approx O(1)$$

$$\tau_{22} \approx -\frac{\mu}{D_1}\Lambda_4\left(\frac{1}{2}\frac{1}{h_3}\frac{\partial u_2}{\partial x_3}\right) \sim O\left(\frac{1}{\delta}\right)$$

$$\tau_{33} \approx +\frac{\mu}{D_1}\Lambda_4\left(\frac{1}{2}\frac{1}{h_3}\frac{\partial u_2}{\partial x_3}\right) \sim O\left(\frac{1}{\delta}\right)$$

$$\tau_{23} \approx -\frac{\mu}{D_1}\Lambda_5\left(\frac{1}{h_3}\frac{\partial u_2}{\partial x_3}\right) \sim O\left(\frac{1}{\delta}\right)$$

$$\tau_{12} \approx -\frac{\mu}{D}\Lambda_7\left(\frac{1}{h_3}\frac{\partial u_1}{\partial x_3}\right) \sim O\left(\frac{1}{\delta}\right)$$

$$\tau_{13} \approx -\frac{\mu}{D}\Lambda_6\left(\frac{1}{h_3}\frac{\partial u_1}{\partial x_3}\right) \sim O\left(\frac{1}{\delta}\right)$$

Furthermore, the divergence of the tensor τ_{ij} in general orthogonal coordinates may be written as

$$\tau_{ij,j} = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial x_j}\left(\frac{h_1h_2h_3}{h_j}\tau_{ij}\right) + \Gamma_{jil}\,\tau_{lj}$$

If it is desired to retain only quantities of the order of $O(1/\delta^2)$, then most of the terms can be ignored except

$$\tau_{33,\,3} \approx -\frac{1}{h_3}\frac{\partial}{\partial x_3}\left(\frac{\mu'}{h_3}\frac{\partial u_2}{\partial x_3}\right)$$

$$\tau_{23,\,3} \approx \frac{1}{h_3}\frac{\partial}{\partial x_3}\left(\frac{\mu''}{h_3}\frac{\partial u_2}{\partial x_3}\right)$$

$$\tau_{13,\,3} \approx \frac{1}{h_3} \frac{\partial}{\partial x_3} \left(\frac{\mu'''}{h_3} \frac{\partial u_1}{\partial x_3} \right) \tag{4}$$

where

$$\mu' = \frac{\mu}{D_1} \frac{\Lambda_4}{2}$$

$$\mu'' = \frac{\mu}{D_1} \Lambda_5 \tag{5}$$

$$\mu''' = \frac{\mu}{D} \Lambda_6$$

In the case of fully ionized gases

$$\mu' = \frac{\mu}{2} \left[\frac{4\omega\tau\sqrt{\gamma}}{(1 + 4\omega^2\tau^2\gamma)} - \frac{4\omega\tau\sqrt{\gamma}}{(1 + \sqrt{2})^2 \left(1 + \dfrac{4\,\omega^2\tau^2}{(1 + \sqrt{2})^2}\right)} \right]$$

$$\mu'' = \mu \left[\frac{1}{\left(1 + 4\omega^2\tau^2\gamma\right)} + \frac{\sqrt{\gamma}}{1 + \sqrt{2}} \frac{1}{\left(1 + 4\,\dfrac{\omega^2\tau^2}{(1 + \sqrt{2})^2}\right)} \right] \tag{6}$$

$$\mu''' = \mu \left[\frac{1}{\left(1 + \omega^2\tau^2\gamma\right)} + \frac{\sqrt{\gamma}}{1 + \sqrt{2}} \frac{1}{\left(1 + \dfrac{\omega^2\tau^2}{(1 + \sqrt{2})^2}\right)} \right]$$

With these results, the momentum and continuity equations can be written in orthogonal curvilinear coordinates. Thus,

$$\frac{\partial \rho}{\partial t} + \frac{1}{h_1 h_2 h_3}\left(\frac{\partial h_2 h_3 \rho u_1}{\partial x_1} + \frac{\partial h_1 h_3 \rho u_2}{\partial x_2} + \frac{\partial h_2 h_1 \rho u_3}{\partial x_3}\right) = 0 \tag{7}$$

$$\frac{\partial u_1}{\partial t} + \frac{u_1}{h_1}\frac{\partial u_1}{\partial x_1} + \frac{u_2}{h_2}\frac{\partial u_1}{\partial x_2} + \frac{u_3}{h_3}\frac{\partial u_1}{\partial x_3} + \frac{u_1 u_2}{h_1 h_2}\frac{\partial h_1}{\partial x_2} - \frac{u_2^2}{h_1 h_2}\frac{\partial u_2}{\partial x_1}$$

$$= -\frac{1}{\rho h_1}\frac{\partial p}{\partial x_1} + F_1 + \frac{1}{\rho h_3}\frac{\partial}{\partial x_3}\left(\frac{\mu'''}{h_3}\frac{\partial u_1}{\partial x_3}\right) \tag{8}$$

and[3]

$$\frac{\partial u_2}{\partial t} + \frac{u_1}{h_1}\frac{\partial u_2}{\partial x_1} + \frac{u_2}{h_2}\frac{\partial u_2}{\partial x_2} + \frac{u_3}{h_3}\frac{\partial u_2}{\partial x_3} + \frac{u_1 u_2}{h_1 h_2}\frac{\partial h_2}{\partial x_1} - \frac{u_1^2}{h_1 h_2}\frac{\partial h_1}{\partial x_2}$$

$$= -\frac{1}{\rho h_2}\frac{\partial p}{\partial x_2} + F_2 + \frac{1}{\rho h_3}\frac{\partial}{\partial x_3}\left(\frac{\mu''}{h_3}\frac{\partial u_2}{\partial x_3}\right) \tag{9}$$

$$\frac{1}{\rho h_3}\frac{\partial p}{\partial x_3} + \frac{1}{\rho h_3}\frac{\partial}{\partial x_3}\left(\frac{\mu'}{h_3}\frac{\partial u_2}{\partial x_3}\right) - F_3 = 0 \tag{10}$$

where F_1, F_2, and F_3 are the components of the ponderomotive force

$$\mathbf{F} = \rho_e\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{J} \times \mathbf{B}$$

where ρ_e is the net charge density, and $\mathbf{J}$ is the conduction current. If $\rho_e = 0$, $\mathbf{F} = \mathbf{J} \times \mathbf{B}$.

[3] It can be postulated that $(\partial u_3/dt)$ is of the order of $O(1)$.

One interesting feature immediately obvious in the present case is that, because of the presence of Hall current, the transverse pressure gradient can no longer be neglected. If the current generated by the diffusion phenomena is assumed to be negligible, then the Ohm's law may be approximated as

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \sigma' \; B^{-1} \; \mathbf{B} \times (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{11}$$

where σ and σ' are the effective conductivities in the two different directions. (If $\omega\tau \approx 1$, $\sigma \approx \sigma'$). From Eq. (11), it can be shown that, if $\rho_e = 0$, $\mathbf{E} = 0$ and the induced magnetic field is small,

$$\begin{aligned} F_2 &= -\sigma u_2 B^2 \\ F_3 &= \sigma' (\mu_2 \mathbf{i}_2 \times \boldsymbol{B}) \cdot \boldsymbol{B} \end{aligned} \tag{12}$$

Thus far it has been assumed that $(\mu_3 \mathbf{i} \times \boldsymbol{B})$ is negligible compared to $(\mu_2 \mathbf{i}_2 \times \boldsymbol{B})$. Before going further, consideration is given to the energy equation, the general form of which may be written as

$$\rho\left(\frac{Dh}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt}\right) + \text{div } \mathbf{Q} = \mathbf{J} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \Phi \tag{13}$$

where $\mathbf{J}$ is the conduction current, Φ is the viscous dissipation function, and $\mathbf{Q}$ is the heat flux. Furthermore, if the total specific enthalpy h_0 defined by $h_0 \equiv h + (1/2)\, v^2$ is introduced, then the energy equation takes the form

$$\rho\left(\frac{Dh_0}{Dt} - \frac{1}{\rho}\frac{\partial p}{\partial t}\right) + \text{div } \mathbf{Q} = (\mathbf{J} + \rho_e) \cdot \mathbf{E} + \nabla(\overline{\overline{\tau}} \cdot \mathbf{v}) \tag{14}$$

In the most general case, the heat flux $\mathbf{Q}$ depends on both thermal conduction and diffusion phenomena. However, in the boundary layer theory, it will be assumed that $\mathbf{Q}$ is determined mainly by thermal conduction.[4] After the boundary layer approximation is employed, the above energy equation becomes

[4] It is important to note that this approximation is good only when no chemical reactions or mass injections take place in the boundary layer region.

$$\rho\left(\frac{\partial h}{\partial t}+\frac{u_1}{h_1}\frac{\partial h}{\partial x_1}+\frac{u_2}{h_2}\frac{\partial h}{\partial x_2}+\frac{u_3}{h_3}\frac{\partial h}{\partial x_3}\right)-\left(\frac{\partial p}{\partial t}+\frac{u_1}{h_1}\frac{\partial p}{\partial x_1}+\frac{u_2}{h_2}\frac{\partial p}{\partial x_2}+\frac{u_3}{h_3}\frac{\partial p}{\partial x_3}\right)$$

$$=\frac{1}{h_3}\frac{\partial}{\partial x_3}\left(\frac{k'}{h_3}\frac{\partial T}{\partial x_3}\right)+\mathbf{J}\cdot(\mathbf{E}+\mathbf{v}\times\mathbf{B})+\mu''\left(\frac{1}{h_3}\frac{\partial u_2}{\partial x_3}\right)^2+\mu'''\left(\frac{1}{h_3}\frac{\partial u_1}{\partial x_3}\right)^2 \tag{15}$$

where k' is the effective thermal conductivity and may be expressed as follows:

$$k' = k_0\,\psi\left(\omega,\tau,\gamma,\frac{n_e}{n},\text{ etc}\right)$$

In general, the function ψ is expected to be smaller than unity for all conditions. The discussion of the function ψ based on kinetic theory will be given in a separate paper. Again, it is seen that in the boundary layer theory the Righi-Leduc effect, which is perpendicular to both **B** and $(1/h_3)(\partial T/\partial h_3)$, does not appear.

In terms of h_0, where now $h_0 = h + (1/2)(u_1^2 + u_2^2)$, the energy equation becomes

$$\rho\left(\frac{\partial h_0}{\partial t}+\frac{u_1}{h_1}\frac{\partial h_0}{\partial x_1}+\frac{u_2}{h_2}\frac{\partial h_0}{\partial x_2}+\frac{u_3}{h_3}\frac{\partial h_0}{\partial x_3}\right)-\frac{\partial p}{\partial t}$$

$$=\frac{1}{h_3}\frac{\partial}{\partial x_3}\left(\frac{k'}{h_3}\frac{\partial T}{\partial x_3}\right)+(\mathbf{J}+\mathbf{v}\rho_e)\cdot\mathbf{E}+\frac{1}{h_3}\frac{\partial}{\partial x_3}\left(\frac{1}{2}\frac{\mu''}{h_3}\frac{\partial u_2^2}{\partial x_3}+\frac{1}{2}\frac{\mu'''}{h_3}\frac{\partial u_1^2}{\partial x_3}\right) \tag{16}$$

However,

$$k'\frac{1}{h_3}\frac{\partial T}{\partial x_3}=\frac{k'}{h_3c_p}\frac{\partial h}{\partial x_3}-\frac{k'}{c_ph_3}\left[\frac{1}{\rho}+\frac{T}{\rho^2}\left(\frac{\partial\rho}{\partial T}\right)_p\right]\frac{\partial p}{\partial x_3}$$

$$=\frac{\mu'''}{Pr}\left\{\frac{1}{h_3}\frac{\partial h_0}{\partial x_3}-\frac{1}{h_3}\frac{\partial}{\partial x_3}\left[\frac{1}{2}(u_1^2)+\frac{1}{2}(u_2)^2\right]\right\}-\frac{\mu'''}{Pr}\frac{1}{h_3}\left[\frac{1}{\rho}+\frac{T}{\rho^2}\left(\frac{\partial\rho}{\partial T}\right)_p\right]\frac{\partial p}{\partial x_3} \tag{17}$$

where

$$Pr = \frac{c_p \mu'''}{k'}$$

Thus

$$\rho\left(\frac{\partial h_0}{\partial t} + \frac{u_1}{h_1}\frac{\partial h_0}{\partial x_1} + \frac{u_2}{h_2}\frac{\partial h_0}{\partial x_2} + \frac{u_3}{h_3}\frac{\partial h_0}{\partial x_3}\right) - \frac{\partial p}{\partial t} = (\mathbf{J} + \rho_e \mathbf{v}) \cdot \mathbf{E} + \frac{1}{h_3}\frac{\partial}{\partial x_3}\left\{\frac{\mu'''}{Pr}\left(\frac{1}{h_3}\frac{\partial h_0}{\partial x_3}\right)\right.$$

$$\left. - \frac{\mu'''}{Prh_3}\left[(1 - Pr)\frac{\partial}{\partial x_3}\left(\frac{u_1^2}{1}\right) + (1 - \alpha Pr)\frac{\partial}{\partial x_3}\left(\frac{u_2^2}{2}\right)\right] - \frac{\mu'''}{Pr}\frac{1}{h_3}\left[\frac{1}{\rho} + \frac{T}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_p\right]\frac{\partial p}{\partial x_3}\right\} \tag{18}$$

where

$$\alpha = \frac{\mu''}{\mu'''}$$

In the conventional boundary layer theory, the last term may be dropped, since $(1/h_3)(\partial p/\partial x_3) = 0$. However, in the present problem this term should be retained unless the gas is perfect, in which case

$$\frac{1}{\rho} + \frac{T}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_p = 0 \tag{19}$$

Again it is seen that if $\mathbf{E} = 0$ the change of h_0 following a moving fluid element does not explicitly depend upon the electromagnetic field, which is to say that the work done by the Lorentz force just balances the dissipation due to Joule heating.

C. Further Approximations

As mentioned previously, the boundary layer theory furnishes considerable mathematical simplification to the present problem. Besides the discussion in the preceding section, another difficulty appearing in the problem may be removed by an approximation consistent with the boundary layer concept. That is, the overall variation of the magnetic field across the boundary

layer may be ignored except in the case of highly conducting fluids. This enables the introduction of the following additional approximations:

1. The effective transport coefficients may be assumed to rely on the imposed magnetic field strength, which is known. This reduces at least one dependent variable in the determination of the local effective transport coefficients of the ionized gas.

2. The ponderomotive force may now be expressed in terms of the imposed magnetic field. This makes it possible to solve the hydrodynamic equations independently without considering Maxwell's field equation at the same time.

The latter approximation may be assumed to be valid also in the unsteady case, but simplification can be obtained only when the imposed field is not time-dependent.

In many physical problems, the net charge density ρ_e is vanishingly small and the electrical field $\mathbf{E}$ also disappears; thus the ponderomotive force $\mathbf{F}_t$ may be written as

$$\mathbf{F}_t = -\sigma \mathbf{v} B_0^2$$

where σ is the electrical conductivity, B_0 is the component of imposed magnetic field normal to the free-stream velocity, and $\mathbf{v}$ is the velocity in the boundary layer parallel to the free stream. The force normal to the wall (F_n) may be written as [5] $\mathbf{F}_n = \sigma' B_0(\mathbf{v} \times \mathbf{B}_0)$.

To summarize the equations discussed previously,

$$\frac{\partial \rho}{\partial t} + \frac{1}{h_1 h_2 h_3}\left(\frac{\partial h_2 h_3 \rho u_1}{\partial x_1} + \frac{\partial h_1 h_3 \rho u_2}{\partial x_2} + \frac{\partial h_1 h_2 \rho u_3}{\partial x_3}\right) = 0 \tag{20}$$

$$\frac{\partial u_1}{\partial t} + \frac{u_1}{h_1}\frac{\partial u_1}{\partial x_1} + \frac{u_2}{h_2}\frac{\partial u_1}{\partial x_2} + \frac{u_3}{h_3}\frac{\partial u_1}{\partial x_3} + \frac{u_1 u_2}{h_1 h_2}\frac{\partial h_1}{\partial x_2} - \frac{u_2^2}{h_1 h_2}\frac{\partial h_2}{\partial x_1}$$

$$= -\frac{1}{\rho h_1}\frac{\partial p}{\partial x_1} + \frac{1}{\rho h_3}\frac{\partial}{\partial x_3}\left(\frac{\mu'''}{h_3}\frac{\partial u_1}{\partial x_3}\right) \tag{21}$$

[5] The justification is that $-\sigma u_3 B_0^2$ is of the order of $O(\delta)$.

$$\frac{\partial u_2}{\partial t} + \frac{u_1}{h_1}\frac{\partial u_2}{\partial x_1} + \frac{u_2}{h_2}\frac{\partial u_2}{\partial x_2} + \frac{u_3}{h_3}\frac{\partial u_3}{\partial x_3} + \frac{u_1 u_2}{h_1 h_2}\frac{\partial h_2}{\partial x_1} - \frac{u_1^2}{h_1 h_2}\frac{\partial h_1}{\partial x_2}$$

$$= -\frac{1}{\rho h_2}\frac{\partial p}{\partial x_2} - \sigma u_2 B_0^2 + \frac{1}{\rho h_3}\frac{\partial}{\partial x_3}\left(\frac{\mu''}{h_3}\frac{\partial u_2}{\partial x_3}\right) \tag{22}$$

$$\frac{1}{h_3}\frac{\partial p}{\partial x_3} + \frac{1}{h_3}\frac{\partial}{\partial x_3}\left(\frac{\mu'}{h_3}\frac{\partial u_2}{\partial x_3}\right) = \sigma'\ \mathbf{B}_0 \cdot (\mathbf{u}_2 \times \mathbf{B}_0) \tag{23}$$

$$\rho\left(\frac{\partial h}{\partial t} + \frac{u_1}{h_1}\frac{\partial h}{\partial x_1} + \frac{u_2}{h_2}\frac{\partial h}{\partial x_2} + \frac{u_3}{h_3}\frac{\partial h}{\partial x_3}\right) - \left(\frac{\partial p}{\partial t} + \frac{u_1}{h_1}\frac{\partial p}{\partial x_1} + \frac{u_2}{h_2}\frac{\partial p}{\partial x_2} + \frac{u_3}{h_3}\frac{\partial p}{\partial x_3}\right)$$

$$= \frac{1}{h_3}\frac{\partial}{\partial x_3}\left(\frac{\mu'''}{Pr h_3}\frac{\partial h}{\partial x_3}\right) + (\sigma + \sigma')\ u_2^2 B_0^2 + \mu''\left(\frac{1}{h_3}\frac{\partial u_2}{\partial x_3}\right)^2 + \mu'''\left(\frac{1}{h_3}\frac{\partial u_1}{\partial x_3}\right)^2$$

$$-\frac{1}{h_3}\frac{\partial}{\partial x_3}\frac{\mu'''}{Pr}\frac{1}{h_3}\left[\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p\right]\frac{\partial p}{\partial x_3} \tag{24}$$

where

$$Pr = \frac{\mu'''}{c_p k'}$$

or

$$\rho\left(\frac{\partial h_0}{\partial t}+\frac{u_1}{h_1}\frac{\partial h_0}{\partial x_1}+\frac{u_2}{h_2}\frac{\partial h_0}{\partial x_2}+\frac{u_3}{h_3}\frac{\partial h_0}{\partial x_3}\right)+\frac{\partial p}{\partial t}$$

$$=\frac{1}{h_3}\frac{\partial}{\partial x_3}\left\{\frac{\mu'''}{Pr}\frac{1}{h_3}\frac{\partial h_0}{\partial x_3}-\frac{\mu'''}{Prh_3}\left[(1-Pr)\frac{\partial}{\partial x_3}\left(\frac{u_1^2}{2}\right)\right.\right.$$

$$\left.\left.+(1-\alpha Pr)\frac{\partial}{\partial x_3}\left(\frac{u_2^2}{2}\right)\right]-\frac{\mu'''}{Prh_3}\left[\frac{1}{\rho}+\frac{T}{\rho^2}\left(\frac{\partial\rho}{\partial T}\right)_p\right]\frac{\partial p}{\partial x_3}\right\} \tag{25}$$

III. REDUCTION OF CONVECTIVE HEAT TRANSFER

Consideration is now given to two limiting cases: (1) $u_1 = 0$, $u_2 \neq 0$ and (2) $u_1 \neq 0$, $u_2 = 0$. In the first case, the magnetic field interacts with both the flow field and the transport coefficients. However, in the second case the interaction between the flow field and the magnetic field becomes negligible. Intuitively, it may be expected that the maximum reduction of heat transfer should occur in case 1, but this is not always true. Another distinct feature which appears in case 1 is that the transverse pressure gradient exists across the viscous boundary layer. This indicates that the pressure on the wall is significantly smaller than the free-stream pressure. This fact may result in further contributions to the "cross-flow" and "separation" phenomena.

In case 2, the term concerning Joule heating vanishes in the energy equation. If the gas is considered to be perfect, then all equations exhibit the features of a system without a magnetic field. The only distinction is in the effective values of viscosity and thermal conductivity. Therefore, the discussions for conventional compressible boundary layer may be considered to be valid. A complete analysis of case 1 is not simple, especially when the gas outside the boundary layer is also ionized. Since the flow field will be distorted by the imposed field, an investigation based on an "inviscid" model must be made of the boundary condition for the boundary layer solution.

Qualitatively, a reduction of heat transfer is expected if thermal conductivity and Prandtl number can be reduced. The effective values of conductivity and Prandtl number depend upon the electron collision time τ and cyclotron frequency ω. It would be preferable to increase the value of τ as much as possible since the optimum value of ω will depend upon the maximum magnetic field available, whereas the optimum value of τ will depend upon the electron collision cross sections and particle number densities of all kinds of species which make up the gas inside the boundary layer. It would appear that τ is generally not "controllable," being determined by the physical environments. However, in recent years, the idea of reducing heat transfer by means of mass injection has been developed and holds some promise. This process may be used in the present case in such a way that the injected gas will have a much smaller electron collision cross section over a certain practical temperature range than other species occurring in the boundary layer. For example, argon has a Maxwellian-averaged collision cross section of the order of 2.5×10^{-17} cm^2 $\rightarrow$ 5×10^{-17} cm^2 over the range 5,000°K $\sim$ 10,000°K; in contrast, the electron collision cross section for N and O over the same temperature range is of the order of 1.2×10^{-15} cm^2 $\rightarrow$ 0.9×10^{-15} cm^2, which is about 50 $\rightarrow$ 20 times larger. In this case the combined effect on reduction of heat transfer rate may be of a quite significant order of magnitude. Of course, the physical situation in this case will be more complicated than that which has been stated here. Since the above statement only manifests the fundamental idea, qualitative results will depend upon further theoretical and experimental investigations.

REFERENCES

1. Chapman, S., and T. G. Cowling. The Mathematical Theory of Non-Uniform Gases, Cambridge University Press, 1958.

2. Neuringer, J. L., and W. McIlroy. "Incompressible Two-Dimensional Stagnation-Point Flow of an Electrically Conducting Viscous Fluid in the Presence of a Magnetic Field," Journal of Aeronautical Sciences, Vol. 25, March, 1958, p. 194.

3. Neuringer, J. L., and I. G. McIlroy. "Hydromagnetic Effects on Stagnation-Point Heat Transfer," Journal of Aeronautical Sciences, Vol. 25, May, 1958, p. 332.

4. Rossow, V. J. "Magnetohydrodynamic Analysis of Heat Transfer Near a Stagnation Point," Journal of Aeronautical Sciences, Vol. 25, May, 1958, p. 334.

5. Bleviss, Z. O. "Magnetogasdynamics of Hypersonic Couette Flow," Journal of Aero/Space Sciences, Vol. 25, October, 1958, p. 601.

6. Schlichting, H. Boundary Layer Theory. New York, N. Y., McGraw-Hill Book Co., 1955.

7. Burgers, J. M. Selected Topics from the Theory of Gas Flow at High Temperatures, Part II, AFOSR TN-48-427a. The Institute for Fluid Dynamics and Applied Physics, University of Maryland, May, 1958.